Num. Methods in CAE – WS 19/20 – Short solutions

Exercise 1 (13 points):

(a) $\boldsymbol{x}^{(1)} = \left(1, -1, \frac{1}{2}, 0\right)$

(b) *No*, since $\mathbf{A}$ is not strictly row diagonal dominant.

(c) $\mathbf{M} = -\begin{pmatrix} 0 & 1/2 & 1/2 & 1/2 \\ 1/4 & 0 & 1/4 & 1/4 \\ 1/8 & 1/8 & 0 & 1/8 \\ 1/16 & 1/16 & 1/16 & 0 \end{pmatrix}$

(d) $\|\boldsymbol{x}^{(n)} - \boldsymbol{x}^*\|_1 \leqslant 20\left(\frac{7}{8}\right)^n$

(e) $\|\boldsymbol{x}^{(10)} - \boldsymbol{x}^*\|_1 < \frac{63}{250}$

(f) *No*, since strict row diagonal dominance of $\mathbf{A}$ is *sufficient*, but not *necessary.*

Exercise 2 (7 points):

(a) $p_2(x) = 1 - \frac{x}{2} + \frac{x^2}{8}$

(b) $I \approx \frac{4}{3}$

Exercise 3 (14 points):

(b) $\mathrm{Re}(c_k) = 0$ for all k and $c_k \sim 1/k$

(c) $c_k = -\frac{j}{\pi k}\begin{cases} 0\,, & k = \pm 4, \pm 8, \pm 12, \dots \\ 1\,, & k = \pm 1, \pm 3, \pm 5, \dots \\ 2\,, & k = \pm 2, \pm 6, \pm 10, \dots \end{cases}$

(d) $T_3(t) = -\frac{j}{\pi}\, e^{jt} + \frac{j}{\pi}\, e^{-jt} - \frac{j}{\pi}\, e^{2jt} + \frac{j}{\pi}\, e^{-2jt} - \frac{j}{3\pi}\, e^{3jt} + \frac{j}{3\pi}\, e^{-3jt}$

(e) $S_f(0) = S_f(\pi/2) = 0, \quad S_f(\pi/4) = 1$

Exercise 4 (12 points):

(a) $w_1 = 2 + \frac{h^2}{3} + \frac{h^2}{6(h^2+1)}\,; \qquad t_1 = h$

(b) $w_1 = 2 + \frac{h^2}{2} - \frac{h^4}{6} + \frac{h^6}{6} - \frac{h^8}{6} \pm \ldots = 2 + \frac{h^2}{2} - \frac{1}{6}\sum_{k=2}^{\infty}(-1)^k h^{2k}$

(c) $y(t) = 2 - \frac{t^2}{2} + O(t^3)$

Exercise 5 (14 points):

(a) $x_1 = 3/2$

(b) $\min\{F(x) : x \in [1,2]\} = F(\sqrt[3]{2}) = \sqrt[3]{2}\,, \quad \max\{F(x) : x \in [1,2]\} = F(2) = 3/2\,,$

$\implies \quad F(x) \in [\sqrt[3]{2}, 3/2] \ \subseteq\ [1,2]$ for $x \in [1,2]$.

(c) *à priori* estimate: Since F' is monotonically increasing on $[1,2]$,

$L = \max\{|F'(x)| : x \in [1,2]\} = \max\{|F'(1)|, |F'(2)|\} = 2/3$ and $n = 21$.

(d) F is contracting, hence Newton's method converges;

$\sqrt[3]{2}$ has multiplicity 1 ($f'(\sqrt[3]{2}) \neq 0$), hence convergence order is at least 2.

(e) Since $F''(x) = 4/x^4 > 0$ and F'' is monotonically decreasing on $[1,2]$,

$M = \max\{|F''(x)/2| : x \in [1,2]\} = F''(1)/2 = 2.$

Quadratic convergence: $|x_{m+1} - \sqrt[3]{2}| \leqslant 2 \cdot 10^{-8}$, $|x_{m+2} - \sqrt[3]{2}| \leqslant 8 \cdot 10^{-16}$.