Hochschule Esslingen – University of Applied Sciences

Master Course Design and Development of Automotive and Mech. Engineering		
Module:	**4201 – Numerical Methods in CAE**	
Semester:	Winter 2019/20	Time: 90 mins
Remarks:	Notes & documents from the lecture; literature No calculator or other electronic devices allowed	

Name ______________ First name ______________ Matr.-Number ______________

Note: 60 points can be achieved in total.

Exercise 1 (13 points):

We consider Jacobi's method for the linear equation system $\mathbf{A} \cdot \boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 8 & 1 \\ 1 & 1 & 1 & 16 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 4 \\ -1 \\ 5 \\ 3 \end{pmatrix}, \qquad \boldsymbol{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

(a) Beginning with $\boldsymbol{x}^{(0)}$, perform one step of Jacobi's method.

(b) Based on the criterion from the lecture: Can we guarantee that Jacobi's method converges? (Give a *short* answer – no lengthy calculations are required here!)

(c) Compute the iteration matrix for Jacobi's method.

(d) Use an appropriate pair of matrix and vector norms and the result from the previous exercise parts to write down an *à priori* estimate for Jacobi's method.

(e) After 10 steps of Jacobi's method you obtain

$$\|\boldsymbol{x}^{(10)} - \boldsymbol{x}^{(9)}\|_1 < \frac{9}{250}, \qquad \|\boldsymbol{x}^{(10)} - \boldsymbol{x}^{(9)}\|_2 < \frac{21}{1000}, \qquad \|\boldsymbol{x}^{(10)} - \boldsymbol{x}^{(9)}\|_\infty < \frac{1}{50}.$$

What can you say about the accuracy of $\boldsymbol{x}^{(10)}$?

(f) According to exercise part **(d)**, Jacobi's method converges. Is this a contradiction to exercise part **(b)**? (Give a *short* argument.)

Exercise 2 (7 points):

(a) Compute the Taylor polynomial p_2 of degree 2 with center point $x_0 = 0$ for

$$f(x) \;:=\; \exp\Big(-\frac{\sin(x)}{2}\Big)\,.$$

(b) Use the result of exercise part **(a)** to compute an approximation for

$$I \;:=\; \int_0^2 f(x)\,dx\,.$$

Exercise 3 (14 points):

We extend the function

$$f_0(t) \;:=\; \begin{cases} 0, & t \in \big[-\pi, -\frac{\pi}{2}\big)\,, \\ -1, & t \in \big[-\frac{\pi}{2}, 0\big)\,, \\ +1, & t \in \big[0, +\frac{\pi}{2}\big)\,, \\ 0, & t \in \big[+\frac{\pi}{2}, +\pi\big) \end{cases}$$

$\boldsymbol{2\pi}$−periodically to a function f.

(a) Sketch the graph of f in the interval $[-2\pi, 2\pi]$.

(b) Considering the function f: Which properties do you expect the complex Fourier coefficients c_k should have?

(c) Compute the complex Fourier coefficients c_k, $k \in \mathbb{Z}$. Give a general form and write down the values for c_k for $-3 \leqslant k \leqslant +3$ explicitly.

(d) Write down the Fourier polynomial T_3 of f.

(e) What are the values $T_f(t)$ of the Fourier series at $t = 0,\ t = \pi/4$ and $t = \pi/2$?

Exercise 4 (12 points):

We want to solve the initial value problem

$$\begin{cases} y'(t) &= \dfrac{t}{y(t)-1} \\ y(0) &= 2 \end{cases}$$

$$\begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{array}$$

using the 3^{rd}–order Runge Kutta method which is defined by the Butcher table depicted to the right and step size h.

(a) Compute the approximation w_1 for the exact solution $y(t_1)$ after the first step. Also write down t_1 as a function of h.

(b) Determine the Taylor series of w_1 with center point 0. *Hint:* Use the geometric series and an appropriate substitution.

(c) Use the results form the previous exercise parts in order to write down all the terms of the Taylor series with center point 0 of the exact solution $y(t)$ which are meaningful.

Exercise 5 (14 points):

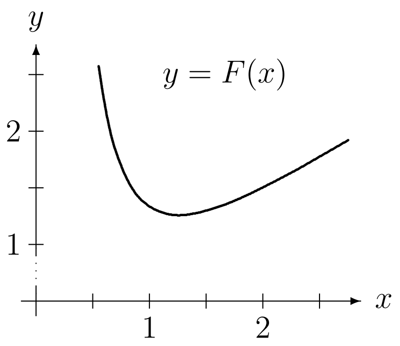
In order to determine an approximation for $x^* := \sqrt[3]{2}$ we apply Newton's method to the function

$$f(x) \; := \; x^3 - 2\,.$$

(a) Beginning with $x_0 := 2$, perform one step of Newton's method in order to compute the approximation x_1.

For the iteration function F and its derivative we have

$$F(x) \; = \; \tfrac{2}{3}\Big(x + \frac{1}{x^2}\Big)\,,$$
$$F'(x) \; = \; \tfrac{2}{3}\Big(1 - \frac{2}{x^3}\Big)\,.$$



(b) Show that F is contracting on $[1,2]$.

Hint: Compute the extremal values of F; you may use the information from the graph of F which is depicted above.

(c) How many steps of Newton's method do you need at most in order to achieve an approximation x_m for which $|x_m - \sqrt[3]{2}| \leqslant 10^{-4}$ holds? (Use $\lg(2) \approx 0.3,\ \lg(3) \approx 0.5$.)

(d) Show that, beginning with $x_0 = 2$, Newton's method converges quadratically to $\sqrt[3]{2}$.

(e) Suppose you have computed an approximation x_m with $|x_m - \sqrt[3]{2}| \leqslant 10^{-4}$. What can you say about the accuracy of the next two approximations x_{m+1}, x_{m+2}?