Hochschule Esslingen – University of Applied Sciences

Master Course Design and Development of Automotive and Mech. Engineering		
Module:	**Numerical Methods in CAE**	Page 1 of 3
Semester:	Winter 2018/19	Time: 90 mins
Remarks:	Notes & documents from the lecture; literature No calculator or other electronic devices allowed	

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Name First name Matr.-Number

Note: 60 points can be achieved in total.

Exercise 1 (7 points):

We want to determine an intersection point of the curves

$$x^4 + (y-1)^4 = 1 \qquad \text{and} \qquad (x-2)^2 + y^2 = 3\,.$$

(a) Rewrite the problem in the form $\boldsymbol{f}(x,y) = \mathbf{0}$ for a function $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$.

(b) Starting with $(x_0, y_0) = (0, 0)$, perform two steps of Newton's method in order to obtain approximations (x_1, y_1) and (x_2, y_2) for one of the zeros of $\boldsymbol{f}$. For both steps, use the Jacobian $\mathbf{J}_f$ at the point (x_0, y_0).

Exercise 2 (11 points):

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} a & 1 & 0 & 0 \\ 1 & a & 1 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 1 & a \end{pmatrix},$$

where a is a real parameter.

(a) For which values of a is $\mathbf{A}$ is strictly row diagonal dominant ?

Let $\boldsymbol{a} = \mathbf{3}$. Furthermore, assume that after having performed k steps of Jacobi's method for solving the LES $\mathbf{A} \cdot \boldsymbol{x} = \boldsymbol{b}$, the last two approximations $\boldsymbol{x}^{(k)}, \boldsymbol{x}^{(k-1)}$ meet the estimate

$$\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\|_\infty \leqslant 10^{-5}\,.$$

(b) What do you know about the accuracy of $\boldsymbol{x}^{(k)}$?

(c) How many additional steps do you need at most until the accuracy of the approximation is better than 10^{-6} ?

Hint: Use $\lg(2) \approx 0.3,\ \lg(3) \approx 0.5$

Exercise 3 (15 points):

The complex Fourier coefficients of

$$f(x) := \begin{cases} 0\,, & -\pi < x \leqslant 0\,, \\ x\,, & 0 < x \leqslant \pi \end{cases}$$

(extended to a $2\pi-$periodic function) are given by

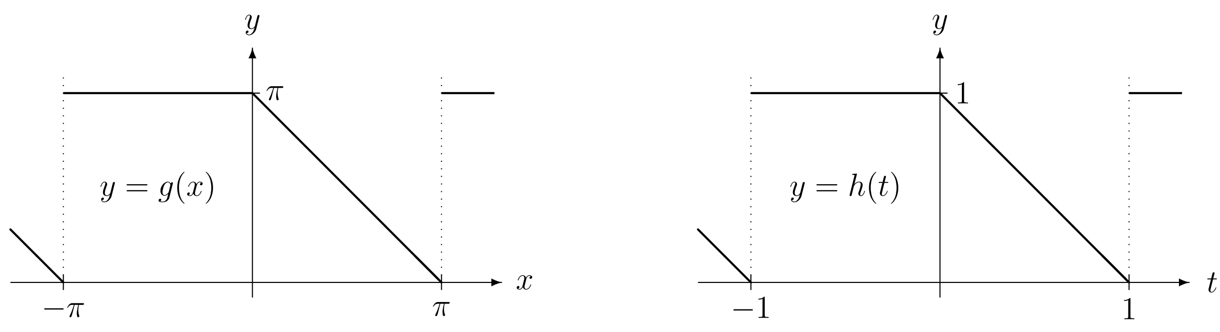
$$c_0 = \frac{\pi}{4}\,, \qquad c_k = \frac{(-1)^k - 1}{2\pi k^2} + j\,\frac{(-1)^k}{2k}\,, \qquad k \in \mathbb{Z}^*\,.$$

(a) Sketch the pointers of $c_0, c_{\pm 1}$ and $c_{\pm 2}$ in the complex plane (use $\pi \approx 3$).

(b) Use c_k to compute the real Fourier coefficients a_k, b_k.

(c) Use the above exercise parts in order to determine the real Fourier coefficients for the functions g and h, the graphs of which are depicted below.

Write down the real Fourier series for g and h up to order $k = 2$ inclusively.



Hint: How are the relations between f, g and g, h, respectively?

Exercise 4 (15 points):

We want to determine an approximation for the integral

$$I \;:=\; \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2(\varphi)}}\,.$$

(a) Compute the Taylor polynomial p_2 at $x_0 = 0$ and Lagrange's remainder term R_2 for

$$f(x) \;=\; \frac{1}{\sqrt{1-x}}\,.$$

(b) Use p_2 of exercise part **(a)** and the integrals below in order to determine an approximation $\tilde{I}$ for the integral I.

(c) Show that for $0 \leqslant x \leqslant 1/2$ Lagrange's remainder term R_2 of exercise part **(a)** meets the estimate

$$\frac{5}{16}\,x^3 \;\leqslant\; R_2(x) \;\leqslant\; \frac{5\sqrt{2}}{2}\,x^3\,.$$

(d) Use the right estimate (estimate for R_2 from above) of exercise part **(c)** for a statement on the accuracy of the approximation $\tilde{I}$.

Hint: $\displaystyle\int_0^{\pi/2} \sin^2(\varphi)\,d\varphi \;=\; \frac{\pi}{4}\,, \qquad \int_0^{\pi/2} \sin^4(\varphi)\,d\varphi \;=\; \frac{3\pi}{16}\,, \qquad \int_0^{\pi/2} \sin^6(\varphi)\,d\varphi \;=\; \frac{5\pi}{32}$

Exercise 5 (12 points):

We consider the initial value problem (IVP)

$$\begin{cases} y'(t) &=\; t + y(t) \\ y(1) &=\; -1 \end{cases}$$

and the Runge Kutta method which is defined by the Butcher table to the right.

$$\begin{array}{c|ccc} \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

(a) Which is the maximal order this Runge Kutta method could have?

(b) Compute the approximation w_1 for the exact value $y(t_1)$ of the solution of the IVP for an arbitrary step size $h > 0$. Also write down t_1 as a function of h.

(c) The exact solution of the IVP reads

$$y(t) \;=\; e^{t-1} - (t+1)\,.$$

Write down the Taylor series for y with center point $t_0 = 1$. *Hint:* Use

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

(d) Let $\tilde{w}_1$ be the approximation to $y(t_1)$ which is obtained from the classical $4^{th}-$order Runge Kutta method. Use the previous exercise parts in order to write down $\tilde{w}_1$.