Num. Methods in CAE – WS 17/18 – Short solutions

Exercise 1 (9 points):

(a) $|a| > 1 + |c| \quad \wedge \quad 12 > |c|$

(b) $a > 1/2 \quad \wedge \quad c^2 < 12a - 6$

(c) **A** is positive definite but *not* strictly diagonal dominant. Hence the SD method is the better choice, since this method must converge; for Gauß-Seidel's method, there is no guarantee for convergence in this case.

(d) $\boldsymbol{x}^{(1)} = \left(0\,,\,0\,,\,\frac{1}{4}\right)$

Exercise 2 (15 points):

$$c_k = \frac{1}{2\pi}\,\frac{1-jk}{1+k^2} \quad (k \in \mathbb{Z})\,; \qquad a_k = \frac{1}{\pi}\,\frac{1}{1+k^2} \quad (k \in \mathbb{N}_0)\,; \qquad b_k = \frac{1}{\pi}\,\frac{k}{1+k^2} \quad (k \in \mathbb{N})$$

$$T_f(t) = \frac{1}{2\pi} + \frac{1}{\pi}\cdot\left(\frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) + \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) + \ldots\right), \qquad \overline{f} = \frac{1}{2\pi}$$

Exercise 3 (10 points):

(a) $w_1 = -1 - \frac{h^2}{2} + \frac{h^3}{3} + \frac{h^5}{6}\,; \qquad t_1 = 1 + h$

(b) $y(t) = -1 - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} + O\big((t-1)^4\big)$

Exercise 4 (8 points):

(a) $p_6(x) = 1 - \frac{3}{2}\,x^2 + \frac{25}{24}\,x^4 - \frac{331}{720}\,x^6$

(b) $\int_0^1 f(x)\,dx \approx \frac{17}{24}$

(c) $\frac{77}{120} < \int_0^1 f(x)\,dx < \frac{31}{40}$

Exercise 5 (15 points):

(a) $f(0) < 0$, $f(\pi/2) > 0$ and f is continuous; furthermore, $f' > 0$ for $x \in [0, \pi/2]$.

(b) $f'(x^*) \neq 0$ shows multiplicity 1. Hence, if we choose an initial value x_0 which is sufficiently close to x^*, Newton's method converges *quadratically*.

(c) $x_1 = \frac{\pi}{4}$

(d) $F(x) = \dfrac{x\sin(x) + \cos(x)}{1 + \sin(x)}$

Analytically for $x \in [0,1]$: From $0 \leqslant x \leqslant 1,\ 0 \leqslant \sin(x),\ 0 \leqslant \cos(x) \leqslant 1$ it follows that $0 \leqslant F(x) \leqslant 1$ on $[0,1]$.

Geometrically from the graph of F: The graph of $F|_I$ is contained in the square $I \times I$.

From the graph of F': $|F'| < 1$ on I. Hence, beginning with an arbitrary value $x_0 \in I$, Newton's method converges to x^*.

(e) $x^* \approx 0.75$

(f) Quadratic convergence. From the picture, $M \lessapprox 0.7$. Hence $|x_{n+1} - x^*| \leqslant 3.5 \cdot 10^{-3}$.