# Hochschule Esslingen – University of Applied Sciences

| Master Course Design and Development of Automotive and Mech. Engineering | | |
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| **Module:** | **Numerical Methods in CAE** | Page 1 of 2 |
| Semester: | Winter 2015/16 | Time: 90 mins |
| Remarks: | Notes & documents from the lecture and 10 pages of personal notes allowed<br>No calculator or other electronic devices allowed | |

Name ______________ First name ______________ Matr.-Number ______________

## Note: 53 points can be achieved in total.

### Exercise 1 (11 points):

We consider the linear equation system $\mathbf{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 1 \\ 0 & a_{22} & -1 \\ 1 & a_{32} & a_{33} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b \\ 0 \\ 1 \end{pmatrix};$$

herein, $a_{ij}, b \in \mathbb{R}$ are real parameters.

**(a)** For which values of $a_{ij}, b$ can we guarantee that the method of the steepest descent (SD) converges? *Hint:* Sylvester's criterion.

**(b)** Use $a_{11} = a_{22} = b = 1,\ a_{32} = -1$ and $a_{33} = 5$. Beginning with $\boldsymbol{x}^{(0)} = (1, 2, 0)$ perform one step of the SD method in order to obtain an approximation $\boldsymbol{x}^{(1)}$ to the exact solution of the LES.

### Exercise 2 (6 points):

We consider the initial value problem

$$y'(t) = -(1 + 2t) \cdot y(t), \quad y(0) = 2.$$

**(a)** Perform one step with the midpoint rule and step size $h$ in order to determine an approximation $w_1$ for the value $y(h)$ of the exact solution.

**(b)** $y(h)$ can be represented by a Taylor series with center point 0 – it then reads

$$y(h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + \ldots, \qquad a_k \in \mathbb{R} \quad \forall\, k \in \mathbb{N}_0.$$

Use the result from exercise part **(a)** in order to determine the values for as many coefficients $a_k$ as possible.

**(c)** How many coefficients could you determine if you had performed a step with the classical $4^{th}$–order Runge-Kutta method instead of the midpoint rule? (Give the number and a *short* argument – no calculation is required!)

**Exercise 3 (11 points):**

We consider the function

$$f(x) := \begin{cases} 0\,, & -\pi < x \leqslant 0\,, \\ x^2\,, & 0 < x \leqslant \pi \end{cases}$$

which is extended to a $2\pi$−periodic function.

**(a)** Compute the complex Fourier coefficients $c_k$, $k \in \mathbb{Z}$ for $f$. Use

$$\int x^2 e^{ax}\, dx \;=\; \frac{1}{a^3}\left(a^2x^2 - 2ax + 2\right) e^{ax} + C$$

**(b)** Use the $c_k$ to compute the real Fourier coefficients $a_k, b_k$ for $f$.

**(c)** Write down the real Fourier series of $f$ up to terms of order $k = 2$ inclusively.

**Exercise 4 (15 points):**

**(a)** Compute the Taylor polynomial $p_2$ of degree 2 at $x_0 = 5$ for

$$f(x) \;=\; \sqrt{x-1}\,.$$

**(b)** Use the result of exercise part **(a)** to determine an approximation for $\sqrt{6}$. *Hint:* For which $x \in \mathbb{R}$ is $f(x) = \sqrt{6}$?

**(c)** Use the remainder term $R_2$ of Lagrange in order to obtain an accuracy statement for the approximation of exercise part **(b)**; i.e. determine an interval $I$ such that $\sqrt{6} \in I$.

**Exercise 5 (10 points):**

In order to determine an approximation to $x^* := \sqrt[7]{a}$ (where $a > 1$) we apply Newton's method to the function

$$f(x) \;:=\; x^7 - a\,.$$

**(a)** Sketch the graph of $f$ for $x > 0$ and give a geometrical argument that Newtons method converges to $x^*$ if we start with $x_0 := a$.

**(b)** Give an argument that Newton's method converges *quadratically* to $x^*$.

**(c)** Compute the iteration function $F$ as well as $F'$ and $F''$.

*Hint:* Using $F'(x) = \dfrac{f(x)f''(x)}{f'(x)^2}$ saves time.

**(d)** Suppose you have computed an approximation $x_n$ with $|x_n - x^*| < 10^{-3}$. Use the quadratic convergence and the above exercise parts in order to give an estimate for the accuracy of $x_{n+1}$ and $x_{n+2}$.

*Hint:* The desired formula must *not* depend on $a$. The estimate $\sqrt[7]{a} > 1$ may be helpful.