Hochschule Esslingen – University of Applied Sciences

Master Course Design and Development of Automotive and Mech. Engineering		
Module:	**Numerical Methods in CAE**	Page 1 of 2
Semester:	Winter 2014/15	Time: 90 mins
Remarks:	Notes & documents from the lecture and 10 pages of personal notes allowed No calculator or other electronic devices allowed	

Name ____________ First name ____________ Matr.-Number ____________

Note: 53 points can be achieved in total.

Exercise 1 (9 points):

We consider the linear equation system $\mathbf{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 6 & -1 \\ 0 & -1 & 5 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix}.$$

(a) Show that the solution of this LES is unique. (It is *not* required to compute the exact solution!)

(b) Can we guarantee that Jacobi's method for solving the LES converges?

(c) Can we also guarantee that the method of the steepest descend converges?

(d) Beginning with $\boldsymbol{x}^{(0)} = (1, 2, 0)$ perform one step of Jacobi's method in order to obtain an approximation $\boldsymbol{x}^{(1)}$ to the exact solution of the LES.

Exercise 2 (12 points):

We consider the function

$$f(t) := e^{-|t|}, \quad t \in [-\pi, \pi) \qquad \text{and extended } 2\pi\text{−periodically,}$$

and we denote the complex Fourier series of f by

$$T_f(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jkt}.$$

(a) Name two properties which you expect the complex Fourier coefficients c_k should have. (Do *not* compute anything but just investigate f.)

(b) Compute the complex Fourier coefficients c_k, $k \in \mathbb{Z}$.

(c) Use c_k to compute the real Fourier coefficients a_k, b_k. Write down the real Fourier series of f up to terms of order $k = 2$ inclusively.

(d) What is the mean value of f?

Exercise 3 (16 points):

In order to determine the intersection point of the curves $y = e^{-x}$ and $y = x$ we apply Newton's method to the function

$$f(x) := x - e^{-x};$$

as initial estimate we choose $x_0 = 0$.

(a) Show that f has exactly one zero x^*, i.e. that the curves intersect exactly once. Show that $x^* \in (0,1)$.

(b) Perform one step of Newton's method in order to obtain an approximation x_1 for x^*.

(c) Show that Newton's method converges. Proceed as follows:

- Compute the iteration function F and simplify the function term such that it contains only e^x, but not e^{-x}.
- Show that F is contracting on the interval $[0,1]$.
- You may assume that F' is strictly monotonically decreasing on $[0,1]$. Use this in order to obtain a suitable estimate for
$$L = \max\{|F'(x)| : x \in [0,1]\}.$$

(d) Show that Newton's method converges quadratically to x^*.

(e) Show that after performing three steps of Newton's method we have $|x_3 - x^*| < 10^{-4}$.
Hint: Do *not* compute x_3, but use
$$\max\{|F''(x)| : x \in [0,1]\} \leqslant \tfrac{1}{2}.$$

Exercise 4 (16 points):

We want to solve the initial value problem

$$\begin{cases} y'(t) = \dfrac{1}{t + y(t)} \\ y(1) = 0 \end{cases}$$

using the 3^{rd}−order Runge Kutta method which is defined by the Butcher table depicted to the right and step size h.

$$\begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{array}$$

(a) Compute the approximation w_1 for the exact solution $y(t_1)$ after the first step. Also write down t_1 as a function of h.

(b) $w_1 = w_1(h)$ is a function of h. How many terms of the Taylor series for w_1 with center point $h = 0$ are meaningful, considering the properties of the Runge Kutta method with which w_1 was derived? Compute the Taylor polynomial of w_1 with the respective degree.
Hint: Geometric series, Cauchy product.

(c) Write down as many terms of the Taylor series with center point 1 of the exact solution $y(t)$ as you know from the result of exercise part **(b)**.