Hochschule Esslingen – University of Applied Sciences

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<tr><td>Semester:</td><td>Winter 2012/13</td><td>Time: 90 mins</td></tr>
<tr><td>Remarks:</td><td colspan="2">Notes & documents from the lecture and 10 pages of personal notes allowed
No calculator or other electronic devices allowed</td></tr>
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Name ____________________ First name ____________________ Matr.-Number ____________

Note: 50 points can be achieved in total.

Exercise 1 (20 points):

We consider the linear equation system $\mathbf{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -4 & 2 \\ 2 & -1 & 4 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}.$$

(a) Show that the solution of this LES is unique. (It is *not* required to compute the exact solution!)

(b) Show that Jacobi's method for solving the LES iteratively converges.

(c) Beginning with $\boldsymbol{x}^{(0)} = (1, 1, 1)$ perform one step of Jacobi's method in order to obtain an approximation $\boldsymbol{x}^{(1)}$ to the exact solution of the LES.

(d) We want to obtain an approximation $\boldsymbol{x}^{(n)}$ which differs by no more than 10^{-4} from the exact solution of the LES in the maximum norm. Determine the number n of iteration steps which are required at most to obtain this accuracy (*à priori* estimate).

Use $\lg(6) \approx 0.8$, $\lg(0.75) \approx -0.12$

Exercise 2 (6 points):

Consider the function

$$\boldsymbol{f}(x, y) := \begin{pmatrix} x^2 - y^2 - 1 \\ (x+2)^2 - y^2 + 1 \end{pmatrix}.$$

(a) Starting with $(x_0, y_0) = (0, 1)$, perform one step of Newton's method in order to obtain an approximation (x_1, y_1) to one of the zeros of $\boldsymbol{f}$. *Hint:* For exercise part **(b)** it is useful to compute $\mathbf{J}_f^{-1}$ for arbitrary values of (x, y).

(b) Does Newton's method converge with the given initial estimate (x_0, y_0)? (Give a short argument.)

Exercise 3 (9 points):

The complex Fourier coefficients for a real periodic function f are given by

$$c_k = \begin{cases} -\dfrac{j}{2k} + \dfrac{1}{\pi k^2}\,, & k = \pm 1, \pm 3, \ldots \\ 1\,, & k = 0\,, \\ \dfrac{j}{2k}\,, & k = \pm 2, \pm 4, \ldots \end{cases}$$

(a) Compute the real Fourier coefficients a_k, b_k of the Fourier series for f.

(b) Suppose the period of f is $T = 2$. Write down the real Fourier series for f up to terms of order $k = 3$ inclusively.

(c) Which properties of f can you derive directly from the Fourier coefficients? (*Hint:* Think about mean value, continuity, symmetry. For each statement give a *short* argument.)

Exercise 4 (15 points):

The initial value problem

$$y' \;=\; \frac{1}{y^2}\,, \qquad y(0) = 1$$

has the solution

$$y(t) = \sqrt[3]{1 + 3t}\,.$$

(a) Compute the Taylor polynomial $p_3(t)$ of degree 3 with center point $t_0 = 0$ for y.

(b) Perform one step of the midpoint rule with step size h in order to determine an approximation w_1 for $y(h)$.

Result: $\quad w_1 \;=\; 1 \;+\; \dfrac{h}{(1 + h/2)^2}$

(c) The approximation w_1 depends on h. Determine the Taylor polynomial $\tilde{p}_3(h)$ of degree 3 with center point $h = 0$ for $w_1(h)$.

Hint: Differentiation of power series or Cauchy product, $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 \pm \ldots$

(d) Explain shortly why the first few terms of the Taylor polynomials of **(a)** and **(c)** agree.

Hint: According to the theory: How many terms *must* agree?